3-1988

Introduction to Supermanifolds

Philip C. Nelson
University of Pennsylvania, nelson@physics.upenn.edu

Recommended Citation

Nelson, P. C. (1988). Introduction to Supermanifolds. *International Journal of Modern Physics A*, 3 (3), 585-590. http://dx.doi.org/10.1142/S0217751X88000230

At the time of publication, author Philip C. Nelson was affiliated with Harvard University. Currently, he is a faculty member in the Physics & Astronomy Department at the University of Pennsylvania.

Introduction to Supermanifolds

Abstract
Talk presented a.t the Workshop on Strings and Riemann Surfaces, Mathematical Sciences Research Institute, Berkeley, March 1986.

Disciplines
Physical Sciences and Mathematics | Physics

Comments
At the time of publication, author Philip C. Nelson was affiliated with Harvard University. Currently, he is a faculty member in the Physics & Astronomy Department at the University of Pennsylvania.

55

HUTP-86/A024

Introduction to Supermanifolds

Philip Nelson

Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138

Talk presented at the Workshop on Strings and Riemann Surfaces, Mathematical Sciences Research Institute, Berkeley, March 1986.

03/86

Introduction to Supermanifolds

Philip Nelson

Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138

1. Introduction

Supermanifolds are the simplest sort of noncommutative extension to ordinary differential geometry. They were originally invented as the natural arena for classical field theories describing fermions. Later they turned out to be useful in conferring a geometrical interpretation upon the "supersymmetries" which some of these theories enjoy. In particular, the supersymmetric string theories which many of the participants will be discussing in this workshop are naturally formulated on supermanifolds with two commuting dimensions and one or two anticommuting ones, in addition to a complex structure – the so-called "super-Riemann surfaces."

Supermanifold theory is a big subject with a long history, and my definitions will not be the most general possible. They do suffice for many applications. The interested reader will find them elaborated in the review [1], along with the history[1] .

We define a smooth **supermanifold** as a pair $\mathcal{X} = (X, A)$, where X is an honest manifold of dimension n and A is a sheaf over X of graded algebras. That is, locally A just associates to each neighborhood of X an algebra in a way which makes sense on overlaps, just as C^∞ associates to any open set the smooth functions thereon. Indeed we will require that locally our

$$A|_U \cong C^\infty(U) \otimes \wedge^\bullet V \quad , \tag{1.1}$$

for some vector space V of dimension m. V may be real or complex. Thus a supermanifold contains the structure of an ordinary manifold, but in addition some extra anticommuting structure. The graded-commutative algebras $A|_U$ will in every respect play the role of $C^\infty(U)$, and so by an abuse of notation we will also refer to A as $C^\infty(\mathcal{X})$. We also say that the dimension of $\mathcal{X}$ is $n|m$.

[1] See also the approaches of [2]

How can we obtain such an A? The easiest way is to give some vector bundle E over X and let $A = \Gamma(\wedge^\bullet E)$, which certainly satisfies (1.1). In fact in the smooth category every supermanifold can be realized in this way, by a theorem of Batchelor, and so we will always specify $\mathcal{X}$ by giving X and E.

To fix ideas we will often return to the example where X is a Riemann surface, perhaps just C, and E is either some square root L of the canonical bundle K over X, or else $L \oplus \overline{L}$. In the former case sections of E are called "chiral spinors," while in the latter they are "nonchiral."

In the rest of this talk I'll give some physical motivation for the above definition, develop it further, and apply it to field theory.

2. Why supermanifolds?

The idea of supermanifolds was already implicitly present in the 1959 paper of J. L. Martin[3]. Martin was looking for a generalization of classical dynamics.

Recall that classical dynamics describes the states of a physical system as points of some **configuration space** M, a manifold. The subject began when Newton described dynamics using a second-order differential equation on M. For those who don't like second-order equations, one can always convert the problem into a first-order system on $\Gamma \equiv T^*M$, the **phase space**. Dynamics then becomes the flow of some vector field V on Γ. Moreover Newton's dynamics comes from a very special V_h, one of the form $V_h = (dh)^t$. Here the adjoint is in the natural symplectic form on Γ and $h \in C^\infty(\Gamma)$ is called the **hamiltonian**.

More generally any function f in $C^\infty(\Gamma)$ is called an **observable**; it reflects some measurable property of the system. For example, h is the energy. All observables correspond to vectors as above, so that $A \equiv C^\infty(\Gamma)$ picks up a Lie algebra structure (the "Poisson brackets") in addition to the usual commutative algebra.

The Lie algebra structure is very important. If V_h defines time evolution then the measured value of any observable evolves as $\dot{f} = V_h f \equiv \{h, f\}$. For this to make sense we need

$$\frac{d}{dt}(fg) = \dot{f}g + f\dot{g}, \tag{2.1}$$

which is ensured by the Jacobi identity.

A key example concerns a particle moving in ordinary $M = \mathrm{R}^3$. Take coordinates q^i for M and q^i, p^i for $\Gamma = T^*M$. Then the dynamics algebra is

$$\{p^i, q^i\} = \delta^{ij}, \qquad \{p^i, p^j\} = \{q^i, q^j\} = 0. \tag{A1}$$

The quantum analog of this system involves representing algebra (A1) on a linear space. I can't discuss quantum mechanics here, but I'll point out two facts. First, the representation space $L_2(\mathrm{R}^3)$ is infinite dimensional. Second, an observable called "angular momentum" has eigenvalues on $L_2(\mathrm{R}^3)$ which are integers.

Particles of integer angular momentum are called **bosons**. In fact, ordinary dynamics *invariably* describes bosons. The problem Martin faced was simply that the world is not made solely of bosons. In addition it contains **fermions**, particles of half-odd angular momentum.

Martin observed that in the foregoing setup the manifold Γ was really logically superfluous. Quantum mechanics discards it altogether, focusing instead on the algebra A of observables. Can we generalize A?

A suitable generalisation preserving the consistency condition (2.1) takes A to be a *graded*-commutative algebra with *graded*-Lie structure. That is,

$$fg = (-)^{|f|\cdot|g|} gf$$

$$(-)^{|f|\cdot|h|}\{f, \{g,h\}\} + \text{cyclic } = 0 . \tag{2.2}$$

If the hamiltonian is even then we still have (2.1). For example, take $A = \wedge^*(\mathrm{C}^3)$ with algebra

$$\{\xi^i, \xi^j\} = \delta^{ij} . \tag{A2}$$

Representing this Clifford algebra gives us a quantum system with angular momentum 1/2: a fermion. Moreover, algebra (A2) has *finite*-dimensional reps, expressing a physical fact called the **Pauli Exclusion Principle**: in contrast to the bosons, only finitely many fermions can "fit" into finitely many states.

3. More definitions

We'd like to combine algebras A1, A2 in a unified, geometrical way, so as better to understand the relations between bosons and fermions. But this is precisely what definition (1.1) does: $C^\infty(\mathcal{X})$ is simply a global version of A1⊗A2. Moreover, since everything in the world is either a boson or a fermion, *that's it*. No more elaborate sheaves than (1.1) are needed.

We now want a notion of vector field. The procedure we use gets repeated over and over in the theory: to generalize an object defined on X to $\mathcal{X}$, formulate it algebraically on $C^\infty(X)$ and then pass to $C^\infty(\mathcal{X})$ with a sprinkling of minus signs. For the case at hand the vector fields of X can be regarded as derivations of $C^\infty(X)$. Similarly we define for $\mathcal{X}$ the sheaf of **graded derivations**, *i.e.* those D with $D(fg) = (Df)g + (-)^{|D|\cdot|f|} fDg$. Commuting we see that vector fields satisfy (2.2).

What's all this look like in coordinates? Take a patch $U \subseteq X$ with coordinates $\{x^i\}$, so that $f \in C^\infty(U)$ may be written as $f = F \circ \vec{x}$. Next choose a frame $\{\theta^\alpha\}$ for the bundle E defining A. In $\wedge^* E$ these satisfy $\theta^\alpha\theta^\beta = -\theta^\beta\theta^\alpha$, so we call them **anticommuting coordinates of** $\mathcal{U} \subseteq \mathcal{X}$. In the full $A|_U$ we further declare that $\theta^\alpha x^i = x^i\theta^\alpha$; thus x^i are **commuting coordinates of** U. A typical $f \in C^\infty(\mathcal{U})$ thus becomes

$$f = F \circ \vec{x} + \theta^\alpha G_\alpha \circ \vec{x} + \theta^\alpha\theta^\beta H_{\alpha\beta} \circ \vec{x} \cdots . \tag{3.1}$$

The finite collection of functions $F, G, H \ldots$ we call **components of** f.

In the example of section 1, with $E = L$, we have simply $f = F(z,\bar{z}) + \theta^+ G_+(z,\bar{z})$. Here the plus reminds that under changes of coordinates G is a spinor, while F is a scalar.

The derivations are now spanned by the symbols $\partial/\partial x^i$, $\partial/\partial\theta^\alpha$ where

$$\frac{\partial}{\partial x^i} x^j \equiv \delta_{ij} \qquad \frac{\partial}{\partial\theta^\alpha}\theta^\beta \equiv \delta_{\alpha\beta}$$

$$\frac{\partial}{\partial\theta^\alpha} x^j \equiv 0 \qquad \frac{\partial}{\partial x^i}\theta^\alpha \equiv 0 .$$

The last may be a surprise, since θ^α is not in general "constant" on U. But this choice is in line with our resolve to treat θ^α on an equal footing with x^i, and a short computation shows that with this rule a vector $V = V^i\partial_i + V^\alpha\partial_\alpha$ has components (V^i, V^α) which transform "like a vector" under changes of coordinates.

In the example, if $V = (a + b\theta)\frac{\partial}{\partial z} + (c + d\theta)\frac{\partial}{\partial\theta}$ then $Vf = (a + b\theta)\partial_z F + a\theta\partial_z G + (c + d\theta)G$.

Sometimes the algebra of vector fields has a distinguished subalgebra. For instance the abstract graded Lie algebra with ξ odd, p even, and

$$[\xi, p] = [p,p] = 0, \quad [\xi,\xi] = 2p \tag{3.2}$$

can be realized as $p \leftrightarrow \partial/\partial\theta$, $\xi \leftrightarrow \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$. A subalgebra picked out as the annihilator of some distinguished function $S \in C^\infty(\mathcal{X})$ is called a **symmetry of** S. If it contains any odd generator it is called **supersymmetry**.

4. Field Theory

Classical **field theory** uses an infinite-dimensional configuration space of maps: $F = \text{Map}\ (X, Y)$. I'll be very casual about the precise definitions of such spaces. For example if X is a Riemann surface and $Y = \mathrm{C}^d$ we have a "theory of d scalar fields on 2-dimensional spacetime." String theory includes such an F. The points $\varphi \in F$ have infinitely many coordinates $\varphi^\mu(x)\quad \mu = 1, \cdots d \quad x \in X$.

Besides a field space F, field theory uses a distinguished function $S \in C^\infty(F)$; for example $S_1[\varphi] = \int_X (\partial\varphi^\mu)(\bar\partial\varphi^\mu)$. S is called the **action**, and it can have symmetries, vector fields on F.

Sometimes a vector V on F arises in a special way, induced from a vector v on X by $V|_\varphi = \int_X (v\varphi)(x) \frac{\delta}{\delta\varphi(x)}\, d(\text{vol})$. For example if $X = \mathrm{C}$ the translation $v = \partial/\partial z$ acts on φ by shifting it. A symmetry which is of this type is called a **spacetime symmetry**; normally it has some geometrical significance on X in that v is an isometry or whatever. In the example S_1 has a spacetime symmetry under translation in C.

So far we've used ordinary commuting spaces. It should not surprise you to learn, then, that classical field theories inevitably describe bosons[2]. What we need is a generalization of F to a supermanifold.

Consider in our example the vector space $B \equiv \Gamma(L^{d'})$ of spinor sections. Its Grassmann algebra $\wedge^\bullet B$ is the odd part of an infinite-dimensional superspace $\mathcal{F}_2$ (whose base is trivial) with anticommuting coordinates $\psi_+^\mu(x)$, $\mu = 1, \cdots d'$ $x \in X$. As our action we can take $S_2[\psi] = \int_X \psi^\mu \bar\partial \psi^\mu$, since the integrand is a $(1,1)$ form. This theory describes fermions. It too has translation symmetry if $X = \mathrm{C}$.

Combining, we get a field superspace $\mathcal{F}$ over F with both φ and ψ, and a translation-symmetric $S = S_1 + S_2$. But now a wonderful thing happens: precisely when $d = d'$ we get a new symmetry, under

$$V_S = \int_{\mathrm{C}} \left[\partial\varphi^\mu \frac{\delta}{\delta\psi^\mu} + \psi^\mu \frac{\delta}{\delta\varphi^\mu}\right]$$

Together with the translation V_T, this supersymmetry furnishes a representation of (3.2):

$$p \leftrightarrow \partial/\partial z \leftrightarrow V_T$$
$$\xi \leftrightarrow \boxed{\ ?\ } \leftrightarrow V_S$$

[2] This is a small cheat thanks to "fermionization" in two dimensions.

5. Supersymmetry

What goes in the box? Why is $d = d'$ so special? To answer these questions we return to our friend the super-Riemann surface $\mathcal{X}$ built on X. We will build from $\mathcal{X}$ a field space $\mathcal{F}$ with $d = d'$, on which the vector $v_s \equiv \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ is what goes in the box.

Consider an *arbitrary* $\mathcal{X}$ and $\mathcal{Y}$. The natural thing to try for Map $(\mathcal{X}, \mathcal{Y})$ is the set Mor $(\mathcal{X}, \mathcal{Y})$ of homomorphisms pulling $C^\infty(\mathcal{Y})$ back to $C^\infty(\mathcal{X})$. But this definition is not functorial; it doesn't satisfy Mor $(\mathcal{X}, \text{Mor}\ (\mathcal{Y}, \mathcal{Z})) = Mor(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$. Besides, Mor $(\mathcal{X}, \mathcal{Y})$ isn't even a superspace.

Instead we try the following. Specialize to $\mathcal{Y} = Y$, an ordinary manifold. Construct the bundles B_i over Map (X, Y) with fibers over a point φ given by

$$B_1|_\varphi = \Gamma(\wedge'_{\text{even}} E \otimes \varphi^* TY)$$
$$B_2|_\varphi = \Gamma(\wedge_{\text{odd}} E \otimes \varphi^* TY)\,.$$

Here $\wedge'$ omits the zeroth power. Let F be the total space of B_1 and $\mathcal{F}$ the superspace with base F, bundle B_2. $\mathcal{F}$ is the **map superspace** Map $(\mathcal{X}, Y)$; it generalizes to arbitrary $\mathcal{Y}$ and is functorial. A useful mnemonic to remember the coordinates of $\mathcal{F}$, their parity and tensor type is the **superfield**

$$\text{"}X^\mu(x,\theta) = \varphi^\mu(x) + \theta^\alpha \psi_\alpha^\mu(x) + \theta^\alpha\theta^\beta F_{\alpha\beta}^\mu(x) + \cdots\text{"}\,; \tag{5.1}$$

the **component fields** φ, ψ, F... alternate in parity.[3] An important feature of this construction is that it treats the zeroth component φ differently from the other even components F.... This is actually well-known from the supersymmetric sigma model, where we say that φ "carries all the topology of Map."

One can readily check that in the example case F becomes just Map (X, C^d) and $\mathcal{F}$ becomes the space in section 4, with $d' = d$. Moreover by manipulating (5.1) we get a natural map from vectors on $\mathcal{X}$ to those on $\mathcal{F}$, under which v_s belongs in the box. Finally an expression like

$$S = \int_{\mathrm{C}} \frac{\partial}{\partial\theta} \langle V_S X^\mu, V_T X^\mu \rangle$$

evidently defines an element in $C^\infty(\mathcal{F})$ which is the S of section 4 and has the full (3.2) as a **spacetime supersymmetry**.

[3] Compare the expansion (3.1) of a *point* of $C^\infty(\mathcal{X})$, where all coefficients were ordinary functions.

From this no-frills introduction one can proceed to define map superspaces with tensor structure on $\mathcal{X}$, eg. "frame superfields" and the like. One can also develop an exterior calculus, starting with the graded vector space dual to the tangent. And one can work out a theory of integration on $\mathcal{X}$ with the right transformation laws. These tools enable one to write superfield actions with symmetries not only under (3.2) but under the full algebra of vectors on $\mathcal{X}$; these are the **supergravity** theories, and they are needed to describe the string[4]
.

Finally the study of super-Riemann surfaces is just beginning. Progress in this direction has been made in [5] and elsewhere.

I would like to thank Josef Bernstein, Raoul Bott, and Is Singer for teaching me some of this material. This work was supported in part by NSF PHY85-15249 and by the Harvard Society of Fellows.

[4] Two-dimensional supergravity is discussed in dozens of papers. The treatment of [4] follows the present line.

References

[1] D. Leites, Usp. Mat. Nauk. **35:I** (1980)3 [= Russ. Math. Surv. **35:I** (1980)1].

[2] A. Rogers, J. Math. Phys. **21**(1980)1352; J. Rabin, L. Crane, Comm. Math. Phys. **100**(1985)141; B. DeWitt, *Supermanifolds*.

[3] J. L. Martin, Proc. Roy. Soc. Lond. **A251** (1959) 536; 543.

[4] P. Nelson, G. Moore, "Heterotic Geometry," preprint.

[5] Yu. Manin, in Arbeitstagung Bonn 1984 (Springer Lecture Notes in Mathematics 1111); "Critical Dimensions of String Theories," preprint; M. Baranov, A. Shvarts, Pisma ZETP **42** (1985)340 [=JETP Letters **42** (1986) 419].

D. Friedan, to appear in Proceedings of the Workshop on String Unification, Santa Barbara 1985; N. Berkovits, Berkeley preprint PTH-85/55; J. Rabin, L. Crane, to appear